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Semisimilar solutions for unsteady free-convective boundary-layer flow on a vertical flat plate

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The analysis of unsteady free convection has classically been made difficult because of the singularities which occur in the governing boundary-layer equations, and because anomalies often occur which are related to the occurrence of these singularities. In the present paper a semisimilar analysis of unsteady free convection in the vicinity of a vertical flat plate is presented, wherein a number of possible wall-temperature variations with time and position are derived. Unique scalings are formulated for the semisimilar equations that aid in the numerical solutions and in the physical interpretation of the results. These scalings collapse the infinite time and position coordinates into a finite region, and present the semisimilar problem in a format bounded by similarity equations. Solutions are carried out which indicate the occurrence of overshoots in the temperature profiles and heat transfer for a variety of conditions. Also, concepts such as the 'limit-of-pure-conduction' and 'leading-edge penetration distance' are shown to require special interpretation under variable wall-temperature conditions.

1. Introduction

Buoyancy-driven flows have become important in recent years as both a distinct category of fluid mechanics and an area of convective heat and mass transfer. Studies of such flows have been carried out in various flow geometries and with various boundary conditions and fluid properties. Historically, analyses were carried out on a boundary-layer basis and mostly addressed applications involving external transport of a steady-state nature. Today, these boundary-layer studies have largely given way to computer simulations of the more general multi-dimensional enclosure flows of steady natural convection. Many problem areas, however, which are uniquely of a classical boundary-layer nature and which are important in applications, as well as in theory, still persist. Free-convective flow which is unsteady is an example of such an area which has yet to be fully described either analytically or computationally, in either full formulation or in the conventional boundary-layer formulation. A unique feature of these external unsteady flows is that sometimes they are similar, sometimes non-similar, and sometimes even non-boundary layer, and their character can change with the time dimension of the problem. In the present paper an analysis is presented which describes some of the unsteady characteristics of one such problem, the classical free-convective boundary-layer flow in the vicinity of a vertical surface.

The basic problem of two dimensional steady-state free convection about a semi-infinite flat plate was first formulated by Schmidt & Beckmann (1930). For this problem, similar solutions to the momentum and energy equations exist. Ostrach (1953) presented numerical solutions of the reduced equations. Wall-temperature variations which yield similar solutions for the steady case were investigated by Finston (1956) and by Yang (1960) and numerical solutions for several of these wall-temperature distributions have been obtained by Sparrow & Gregg (1956, 1958).

When the assumption of steady-state wall-temperature distribution (steady flow) is removed, the complexity of the problem increases dramatically. Not only is the mathematical problem made more difficult by the introduction of a third independent variable (time), but the physics of the problem is complicated by a dual behaviour of the temperature and velocity fields. Consider the case in which a semi-infinite plate is heated impulsively to a constant temperature higher than that of the surrounding medium. At a given point on the plate, during the early phase of the flow development, the temperature and velocity fields develop as if the plate were infinite in extent for there is no mechanism by which the presence of the leading edge can be transmitted instantaneously up the plate. Illingworth (1950) has shown that in the case of an impulsively heated infinite plate the temperature field develops in the same manner as the temperature field in a semi-infinite solid whose surface is impulsively heated. That is, for the infinite plate, the heat transfer is by one-dimensional conduction only. The velocity field develops in a manner analogous to the development of the velocity field over an impulsively set into motion flat plate (the Rayleigh problem).

Once the signal from the leading edge reaches a given position, the flow and heat transfer at that position begin a transition ending in the steady state which is described mathematically by the well-known similar solution of Schmidt & Beckmann (1930). If the signal from the leading edge does not travel fast enough, a very interesting phenomenon may occur. The boundary-layer thickness due to the one-dimensional conduction may become thicker than the steady-state value it will reach at a later time. This causes a minimum in the heat transfer coefficient which will subsequently approach the steady-state value from below.

Virtually all the investigations of transient free convection at the surface of a flat plate have considered the case in which the plate is impulsively heated to a constant temperature. Despite repeated efforts, however, no completely accurate analytical or numerical solution to this problem has been found. Early numerical transient solutions were published by Hellums & Churchill (1962) for the impulsively heated constant-temperature flat plate and Callahen & Marner (1976) for the same problem with mass transfer. These authors used the same method of solution in which the governing equations are solved in three independent variables as an initial-value problem using finite differences. Both of these solutions indicate a minimum in the heat transfer coefficient, but are suspect because they indicate a departure from the one-dimensional conduction solution well before the critical time given theoretically by Nanbu (1971) and generally accepted as accurate. Indeed, in a recent paper Ingham (1978) suggests strongly that difficulties exist in these numerical solutions. Ingham reproduced the method used by the above authors and reproduced their results for the same finite difference step size. However, as the step size was reduced the results were found to diverge.

In more recent numerical studies it was thought that the impulsively heated plate problem of free convection would be analogous to the impulsively started flat plate problem of hydrodynamic boundary-layer theory studied by Stewartson (1951), Dennis (1972) and Hall (1969). However, all attempts to use the numerical methods which work for the impulsively started flat-plate problem failed when applied to the impulsively heated plate problem. Numerical solutions have recently been attempted by the present authors (unpublished work), by Walker, Abbott & Yau (1975), and by Ingham (1977) but to no avail. All of these attempts used proven numerical techniques satisfactory for the impulsively started flat-plate problem. Ingham also points out that Dennis, Elliot, and Brown & Riley, have all attempted numerical solutions in unpublished works, without success. A very interesting technique of treating singular parabolic problems was recently presented by Wang (1983, 1985), in which numerical procedures commonly employed for elliptic equations were adapted to singular parabolic equations. Such a solution technique, however, has not been applied to the coupled equations of free convection.

The only analytical results for the impulsively heated transient flat-plate problem are given by Brown & Riley (1973). They obtained series solutions valid for small or large values of the similarity variable $\tau = t/x^{\frac{1}{2}}$. No solution was obtained for intermediate values of τ where a minimum in the heat transfer coefficient would occur.

A number of investigators have solved transient free convection problems using approximate methods. Sugawara & Michiyoshi (1951) use the method of successive approximation but the second approximation was only carried out for short times. Siegel (1961) used the method of characteristics to solve the transient free convection equations in the von Kármán–Poulhausen integral form and obtained a solution which clearly indicates a minimum in the heat transfer coefficient. Siegel's solution, however, indicates a much shorter limit of pure conduction than is usually accepted as accurate. Gebhart (1961), also using an integral method, treats the more realistic condition of a plate with arbitrary thermal capacity. His results, however, indicate no minimum of the heat transfer coefficient for a plate with an impulsively applied constant heat flux at the surface and no thermal capacity. Finally, Heinisch, Viskanta & Singer (1969) reduced the number of independent variables in the equations governing transient free convection by successive integration. The resulting ordinary differential equations were solved for both the impulsive temperature and heat flux cases. The solutions indicated a minimum of the heat transfer coefficient, but instabilities occurred in the transient region.

In summary, while the analysis of free convection has developed to some extent in recent years, many fundamental aspects of transient flow characteristics still remain to be described. In the classical cases of the semi-infinite vertical plate heated impulsively to a uniform temperature and impulsively heated at constant and uniform heat flux, developments of reliable and complete solutions have been impeded because of the essential singularities associated with the leading edge. Additionally, these mathematical and numerical difficulties have carried over as impediments in the development of other transient free convection solutions. Data on such additional questions as the effects of variable wall temperature and streamwise conduction in unsteady flow, in particular, are not yet available for this geometry.

In the present work the more general case of a semi-infinite flat plate for which the wall temperature varies with time or with position or with both time and position is addressed. The mathematical technique used in the present analysis is the method of semi-similar solutions, in which the number of independent variables is reduced from three to two by an appropriate scaling. The resulting transformed equations are then solved by standard numerical methods developed for solving problems in two independent variables. Using this technique, complete transient solutions are obtained for certain cases where the wall temperature varies with position. All of these transient solutions exhibit a minimum in the heat transfer coefficient before reaching the steady state. Complete unsteady solutions are also obtained for a class of wall temperatures varying with time and position. Partial solutions are found for certain other wall-temperature variations, and the numerical difficulties with these solutions are traced to a singularity in the equations introduced by infinite wall-temperature derivatives at the leading edge of the plate. The results are believed to be benchmarks which potentially are of importance in future numerical studies.

2. Analysis

The heat transfer and fluid motion that occurs in the fluid adjacent to a semi-infinite, vertical flat plate which is heated in an unsteady manner is of primary interest. The fluid heat transfer and motion is described in a rectangular coordinate system attached to the plate such that the $\bar{x}$-axis lies along the plate surface and the $\bar{y}$-axis is normal to the plate. The $\bar{x}$ and $\bar{y}$ components of velocity are $\bar{u}$ and $\bar{v}$, respectively, and the fluid temperature is $\bar{T}$. It is assumed that the difference between the plate temperature and the temperature of the surrounding medium is everywhere small, the heating due to viscous dissipation can be neglected, the fluid can be considered incompressible except that the changes in density are important in producing buoyancy forces, the Boussinesq approximations are valid, and the kinematic viscosity, ν, and the thermal diffusivity, α, may be taken as constant. Under these circumstances the equations of continuity, x momentum, and energy, in the thin viscous layer adjacent to the wall become

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{1}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = g_0\beta(\bar{T}-\bar{T}_\infty)+\bar{\nu}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{2}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \alpha\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}. \tag{3}$$

Here, as usual, $\bar{t}$ is time, g_0 is the acceleration due to gravity, β is the coefficient of thermal expansion of the fluid and $\bar{T}_\infty$ is the temperature of the surrounding fluid (at a large distance from the plate). The boundary conditions are

$$\bar{u}(\bar{x},0,\bar{t}) = \bar{v}(\bar{x},0,\bar{t}) = \lim_{\bar{y}\to 0}\bar{u}(\bar{x},\bar{y},\bar{t}) = 0,$$

$$\bar{T}(\bar{x},0,\bar{t}) = \bar{T}_w(\bar{x},\bar{t}) \text{ for } \bar{t} > 0,$$

$$\lim_{\bar{y}\to\infty}\bar{T}(\bar{x},\bar{y},\bar{t}) = \bar{T}_\infty.$$

Various initial conditions will result from the semi-similar analysis.

It is noted that the velocity boundary conditions are homogeneous and thus indicate no reference velocity. Ostrach (1953) suggests the characteristic velocity

$$U = [g_0\beta L(\bar{T}_{\text{ref}}-\bar{T}_\infty)]^{\frac{1}{2}}.$$

Non-dimensional Grashof and Prandtl numbers are defined respectively by

$$Gr = \beta g_0 L^3(\overline{T}_{\text{ref}}-\overline{T}_{\infty})/\nu^2,$$
$$Pr = \mu\bar{c}/\bar{k},$$

where L is some arbitrary reference length, $\overline{T}_{\text{ref}}$ is some arbitrary reference temperature, μ is the absolute viscosity, $\bar{c}$ is the specific heat and $\bar{k}$ is the thermal conductivity. The following non-dimensional variables are now introduced:

$$x = \frac{\bar{x}}{L}, \quad y = \frac{(Gr)^{\frac{1}{4}}\bar{y}}{L}, \quad t = \frac{Ut}{L},$$
$$\Delta T = \frac{(\overline{T}-\overline{T}_{\infty})}{(\overline{T}_{\text{ref}}-\overline{T}_{\infty})}, \quad u = \frac{\bar{u}}{U},$$
$$v = (Gr)^{\frac{1}{4}}\bar{v}/U.$$

In terms of these non-dimensional variables the equations of motion and energy, (1) to (3) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{4}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \Delta T+\frac{\partial^2 u}{\partial y^2}, \tag{5}$$

$$\frac{\partial \Delta T}{\partial t}+u\frac{\partial \Delta T}{\partial x}+v\frac{\partial \Delta T}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \Delta T}{\partial y^2}, \tag{6}$$

and the boundary conditions become

$$u(x,0,t) = v(x,0,t) = 0, \tag{7}$$

$$\lim_{y\to\infty} u(x,y,t) = 0,$$

$$\Delta T(x,0,t) = \Delta T_{\text{w}}(x,t) \quad (t>0),$$

$$\lim_{y\to\infty} \Delta T(x,y,t) = 0.$$

Equations (4) to (6) are a coupled set of second order, nonlinear partial differential equations in three independent variables. The techniques presently available for solving coupled equations in three independent variables are quite limited. Here, the technique of semi-similar solutions is employed. In semi-similar solutions the number of independent variables is reduced from three to two by an appropriate scaling. This allows the use of fast and accurate numerical techniques in the solution of the reduced equations. The use of the method of semi-similar solutions is not without some penalty. Not all wall-temperature distributions, $\Delta T_{\text{w}}(x,t)$, lend themselves to semi-similar solutions, as will be seen shortly. Thus, solutions to the problem posed above can be obtained only for a limited number of classes of wall-temperature distributions. Nevertheless, investigation of these wall-temperature distributions for which semi-similar solutions are possible yields new information on the physics of the complicated flows of unsteady free convective heat transfer, as well as accurate results which can be used for limit checking more complicated and unproven numerical procedures.

In order to obtain the required reduction in the number of independent variables, we introduce the new scaled coordinates

$$\eta = y/g(x,t), \quad \xi = \xi(x,t),$$

and a non-dimensional stream function $f(\xi,\eta)$ defined by

$$\psi(x,y,t) = g(x,t)\,k(x,t)\,f(\xi,\eta),$$

where the stream function ψ is related to the fluid velocity components in the usual manner, i.e.

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}.$$

In addition, a new dimensionless temperature, defined by

$$\theta = \Delta T/\Delta T_{\mathrm{w}}$$

is introduced. The functions $g(x,t)$ and $\xi(x,t)$ in the new scaled coordinates and the function $k(x,t)$ in the definition of the stream function are, at this point, not specified.

The continuity equation is identically satisfied by the introduction of a stream function. In terms of the new variables, the momentum and energy equations become

$$\frac{\partial^3 f}{\partial\eta^3} + (d+e)f\frac{\partial^2 f}{\partial\eta^2} - d\left(\frac{\partial f}{\partial\eta}\right)^2 - a\frac{\partial f}{\partial\eta} + b\frac{\eta}{2}\frac{\partial^2 f}{\partial\eta^2} - c\frac{\partial^2 f}{\partial\eta\,\partial\xi} + h\left\{\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial\xi}\right\} + j\theta = 0, \quad (8)$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2} + (d+e)f\frac{\partial\theta}{\partial\eta} + \frac{b\eta}{2}\frac{\partial\theta}{\partial\eta} - c\frac{\partial\theta}{\partial\xi} + h\left\{\frac{\partial f}{\partial\xi}\frac{\partial\theta}{\partial\eta} - \frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\xi}\right\} - l\theta - m\frac{\partial f}{\partial\eta}\theta = 0, \quad (9)$$

where

$$a = \frac{g^2}{k}\frac{\partial k}{\partial t}, \quad b = \frac{\partial g^2}{\partial t}, \quad c = g^2\frac{\partial\xi}{\partial t}, \quad d = g^2\frac{\partial k}{\partial x}, \quad (10a\text{–}d)$$

$$e = \frac{k}{2}\frac{\partial g^2}{\partial x}, \quad h = g^2 k\frac{\partial\xi}{\partial x}, \quad j = \frac{g^2}{k}\Delta T_{\mathrm{w}}, \quad (10e\text{–}g)$$

$$l = \frac{g^2}{\Delta T_{\mathrm{w}}}\frac{\partial\Delta T_{\mathrm{w}}}{\partial t}, \quad m = \frac{kg^2}{\Delta T_w}\frac{\partial\Delta T_{\mathrm{w}}}{\partial x}. \quad (10h,i)$$

The boundary conditions, written in terms of the new variables are

$$\left.\begin{aligned} & f(\xi,0) = \frac{\partial f}{\partial\eta}(\xi,0) = 0, \\ & \lim_{\eta\to\infty}\frac{\partial f}{\partial\eta}(\xi,\eta) = 0, \\ & \theta(\xi,0) = 1, \quad \lim_{\eta\to\infty}\theta(\xi,\eta) = 0. \end{aligned}\right\} \quad (11)$$

If semi-similar solutions are to exist and the problem in the new coordinate system is to be a problem in only two independent variables (ξ,η), then the coefficients defined in (10) must be functions of ξ alone. Clearly this puts some limitation not only on the scaling functions $g(x,t)$ and $k(x,t)$, but also on the wall-temperature distribution ΔT_{w}.

Four equations relating the nine coefficients defined in (10) are obtained on the basis of the continuity of the second derivatives of g, ξ, k and ΔT_w with respect to x and to t. These equations are

$$\left.\begin{aligned}(a+b)\,d &= 2ae+cd'-a'h,\\ 2ae &= 2ce'-hb',\\ h(a+b+c') &= c(h'+2e),\\ l'h-2le &= m'c-m(a+b),\end{aligned}\right\} \qquad (12)$$

where primes here denote differentiation with respect to ξ. We note that the analysis to this point is analogous to that of Williams & Johnson (1974).

It would be desirable to use a direct approach to solve this problem, an approach in which ΔT_w was specified and the scaling functions are obtained from solutions of (10) in conjunction with (12). Such a direct approach is not possible, indeed not all possible functions for ΔT_w yield semisimilar solutions. It is necessary then to use an indirect approach to determine the ΔT_ws for which semisimilar solutions are possible and the scaling functions appropriate to each of these functions. Three cases are considered separately in which (i) ΔT_w varies with time alone, (ii) ΔT_w varies with distance alone and (iii) ΔT_w varies with both time and distance.

(i) *Wall temperature varying with time alone*

For this special case clearly the coefficient m is zero. Forming the ratio l/j and solving for k yields

$$k = \frac{l(\Delta T_w)^2}{j(\partial \Delta T_w/\partial t)}. \qquad (13)$$

Introducing this result into (10a) one finds that the ratio

$$\frac{\Delta T_w}{(\partial \Delta T_w/\partial t)^2}\,\frac{\partial^2 \Delta T_w}{\partial t^2}$$

must, at most, be constant. Setting this ratio equal to a constant and solving for ΔT_w results in two solutions

$$\Delta T_w = (c_1 t + c_2)^r, \qquad (14)$$

$$\Delta T_w = k_1\, e^{k_2 t}, \qquad (15)$$

where c_1, c_2, k_1, k_2 and r are arbitrary constants. The correct functional forms for ξ, k and g must now be determined for each of these wall-temperature distributions.

Turning attention first to the wall-temperature distribution given by (14), it is noted from (13) that since l and j are both functions of ξ above, k must be given by

$$k = (c_1 t + c_2)^{r+1} F_1(\xi), \qquad (16)$$

where $F_1(\xi)$ is at this point an arbitrary function of ξ. From (10g) or (10h) one also obtains

$$g^2 = (c_1 t + c_2)\, F_2(\xi), \qquad (17)$$

where F_2 is also an arbitrary function of ξ.

To determine the functional form of ξ we note that the slopes of the lines of constant ξ in the (x, t)-plane are given by

$$\left.\frac{dx}{dt}\right|_{\xi=\text{constant}} = -\frac{\partial \xi/\partial t}{\partial \xi/\partial x} = -\frac{c(\xi)}{h(\xi)}\,k.$$

Since c and h are functions of ξ alone they must be constant along lines of constant ξ. From (13), we obtain the relationship

$$x/(c_1 t+c_2)^{r+2} = F_3(\xi), \tag{18}$$

where F_3 is again an arbitrary function of ξ. This equation, then, gives the functional combination of x and t in ξ. Equations (16), (17) and (18) now specify g, k and ξ to within the same arbitrary functions of a known combination of x and t.

The arbitrary function of ξ, $F_3(\xi)$, is now chosen to give the most meaningful and computationally convenient form of (8) and (9). We note that in the physical problem the range of both the x and the t variables is from zero to infinity. Computationally it would be convenient to collapse this infinite range into some finite interval. If ξ is chosen to be

$$\xi = 1-\exp\left\{\frac{-4x^{\frac{1}{2}}}{(c_1 t+c_2)^{\frac{1}{2}(r+2)}}\right\}, \tag{19}$$

the range of ξ becomes zero to unity for real variations of both x and t.

The arbitrary functions $F_1(\xi)$ and $F_2(\xi)$ in the definitions of k and g^2 (equations (16) and (17)) are now chosen in such a manner that (8) and (9) reduce to familiar similarity equations at the end points of ξ. If this is done, (8) and (9) become more easily interpreted physically, since the similarity equations are valid for known physical conditions. If $F_1(\xi)$ and $F_2(\xi)$ are chosen by inspection to be ξ and 4ξ, respectively, g^2 and k become

$$g^2 = 4[c_1 t+c_2]\,\xi,$$
$$k = [c_1 t+c_2]^{r+1}\,\xi.$$

The constants in k and g^2 have been chosen by experiment to keep the boundary layer essentially the same thickness, in the η-plane, throughout the range of ξ.

With k, g^2 and ξ now known, for this case, the coefficients $a(\xi)$, $b(\xi)$, $c(\xi)$, $d(\xi)$, $e(\xi)$, $h(\xi)$, $j(\xi)$, $l(\xi)$ and $m(\xi)$ can be determined so that the coefficients in the transformed equations of motion and energy are all known and the problem is now completely specified. It is easily shown that the resulting coefficients satisfy the auxiliary equations (12). It is interesting to note that the constant c_2 in (14) for the wall temperature does not appear in the equations of motion. It is found that c_2 must be equal to zero to give the correct initial condition (see below) for the problem. The constant c_1 may also be scaled out of the problem if it is non-zero, but is retained in order to examine the case when it is zero.

For $\xi = 0$, (8) and (9) reduce respectively to the steady state type similarity equations

$$\left.\begin{aligned}\frac{\partial^3 f}{\partial\eta^3}+48f\frac{\partial^2 f}{\partial\eta^2}-32\left(\frac{\partial f}{\partial\eta}\right)^2+4\theta &= 0,\\ \frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2}+48f\frac{\partial\theta}{\partial\eta} &= 0.\end{aligned}\right\} \tag{20}$$

For $\xi = 1$, (8) and (9) reduce respectively to the one-dimensional conduction equation

$$\left.\begin{aligned}\frac{\partial^3 f}{\partial\eta^3}+2c_1\eta\frac{\partial^2 f}{\partial\eta^2}-4(r+1)c_1\frac{\partial f}{\partial\eta}+4\theta &= 0,\\ \frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2}+2c_1\eta\frac{\partial\theta}{\partial\eta}-4rc_1\theta &= 0.\end{aligned}\right\} \tag{21}$$

One of these sets of similarity equations will serve as the initial condition for (8) and (9).

Now consider the problem from a physical standpoint. The non-dimensional velocity in the viscous layer is given, in this case, by

$$u = [c_1 t + c_2]^{r+1} \xi \frac{\partial f}{\partial \eta}.$$

This velocity should be equal to zero for $t = 0$ at all x, and $x = 0$ for all t. These conditions are satisfied only if $c_2 = 0$. Thus, c_2 is taken as zero for the remainder of the analysis.

In the case in which ΔT_w varies exponentially with time, (15), one can follow the same procedure outlined above to find the appropriate functions $\xi(x,t)$, $g(x,t)$ and $k(x,t)$ together with the similarity equations which are appropriate at the end point of the range for ξ. These results are presented by Rhyne (1978). In this case, however, ΔT_w becomes zero only as $t \to -\infty$. This case does not appear to have great physical significance and thus is not treated further here.

(ii) *Wall temperature varying with position alone*

For the case in which the wall temperature varies with position alone, the coefficient l (equation (10*h*)) must be zero. Forming the ratio m/j and solving for k yields

$$k^2 = \frac{m(\Delta T_w)^2}{j(\partial \Delta T_w / \partial x)}. \tag{22}$$

Introducing the result into (10*d*) and making use of (10*g*), one finds that the ratio,

$$\frac{\Delta T_w}{(\partial \Delta T_w/\partial x)^2} \frac{\partial^2 \Delta T_w}{\partial x^2},$$

must, at most, be a constant. Setting this ratio equal to a constant, and integrating twice yields the two possible solutions

$$\Delta T_w = (c_3 x + c_4)^n, \tag{23}$$

$$\Delta T_w = k_3 \, e^{k_4 x}, \tag{24}$$

were c_3, c_4, k_3 and k_4 are constants. Using the same procedure as was used in the previous case it is determined that, for the first of these cases (equation (23)) the functions $\xi(x,t)$, $g(x,t)$ and $k(x,t)$ are

$$\xi = 1 - \exp\{-\tfrac{1}{4}t(c_3 x + c_4)^{\frac{1}{2}(n-1)}\}, \tag{25}$$

$$g^2 = 16[c_3 x + c_4]^{\frac{1}{2}(1-n)} \xi, \tag{26}$$

$$k = 4[c_3 x + c_4]^{\frac{1}{2}(1+n)} \xi. \tag{27}$$

As before, k, g^2, and ξ are known for this case and the coefficients $a(\xi)$, $b(\xi)$, $c(\xi)$, $d(\xi)$, $e(\xi)$, $h(\xi)$, $j(\xi)$, $l(\xi)$ and $m(\xi)$ are easily determined.

For $\xi = 0$, (8) and (9) reduce respectively to the classical one-dimensional conduction case

$$\left.\begin{aligned} \frac{\partial^3 f}{\partial \eta^3} + 2\eta \frac{\partial^2 f}{\partial \eta^2} - 4\frac{\partial f}{\partial \eta} + 4\theta = 0, \\ \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} + 2\eta \frac{\partial \theta}{\partial \eta} = 0. \end{aligned}\right\} \tag{28}$$

At $\xi = 1$, a separate set of steady-state similarity equations exist. These are

$$\left.\begin{aligned}\frac{\partial^3 f}{\partial\eta^3}+16c_3(n+3)f\frac{\partial^2 f}{\partial\eta^2}-32c_3(n+1)\left(\frac{\partial f}{\partial\eta}\right)^2+4\theta = 0,\\ \frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2}+16c_3(n+3)f\frac{\partial\theta}{\partial\eta}+64c_3 n\theta\frac{\partial f}{\partial\eta} = 0.\end{aligned}\right\} \tag{29}$$

The non-dimensional velocity in the viscous layer, in this case, is

$$u = 4[c_3 x + c_4]^{\frac{1}{2}(1+n)}\,\xi f'(\xi).$$

Again, the velocity should be zero for $t = 0$ at all x and at $x = 0$ for all t, and again these conditions are satisfied only if $c_4 = 0$. Thus c_4 is taken as zero as before for the remainder of the analysis of this case.

When ΔT_w varies exponentially with distance (equation (24)) the same procedure outlined above can be followed to find the functions $\xi(x,t)$, $g^2(x,t)$ and $k(x,t)$, together with the similarity equations at the end points of the range for ξ. These results are also presented by Rhyne (1978). In this case, however, ΔT_w becomes zero only as $x \to -\infty$. This case can be interpreted as the transient free convection near an infinite plate with an exponential growth in wall temperature for $x > 0$ and an exponential decay in temperature for $x < 0$. The corresponding steady-state problem has been studied by Gebhart & Mollendorf (1969). Because of its limited physical significance, this case is not treated further here.

(iii) *Wall temperature varying with position and time*

Finally, the general case in which the wall temperature varies with both position and time is considered. In this case, the expression for k in terms of the ratio of m to j becomes

$$k^2 = \frac{m}{j}\frac{(\Delta T_w)^2}{(\partial\Delta T_w/\partial x)}.$$

On the other hand, if we form the ratio l/j and solve for k, we obtain

$$k = \frac{l}{j}\frac{(\Delta T_w)^2}{(\partial\Delta T_w/\partial t)}.$$

Introducing the first of these into the expression for $d(\xi)$ (equation (10d)), one finds that

$$\frac{\Delta T_w}{(\partial\Delta T_w/\partial x)^2}\frac{\partial^2\Delta T_w}{\partial x^2} = \text{constant}. \tag{30}$$

Introducing the second of these into the expression for $a(\xi)$ (equation (10a)) yields

$$\frac{\Delta T_w}{(\partial\Delta T_w/\partial t)^2}\frac{\partial^2\Delta T_w}{\partial t^2} = \text{constant}. \tag{31}$$

In addition, introducing the first of the above expressions into (10a) or the second into (10d) yields

$$\frac{\Delta T_w}{(\partial\Delta T_w/\partial t)(\partial\Delta T_w/\partial x)}\frac{\partial^2\Delta T_w}{\partial x\,\partial t} = \text{constant}. \tag{32}$$

There are three possible wall-temperature distributions which simultaneously satisfy (30), (31) and (32). These are

$$\Delta T_w = (c_1 t + c_2)^r\,(c_3 x + c_4)^n, \tag{33}$$

$$\Delta T_w = k_1\,e^{k_2 t + k_4 x}, \tag{34}$$

and
$$\Delta T_w = [c_1 x + c_2 t]^n \tag{35}$$
where c_1, c_2, c_3, c_4, k_1, k_2 and k_4 are constant. It is noted that the cases discussed earlier are actually special cases of either (33) or (34).

Following a procedure which is similar to, but more involved than, that in the previous section it can be shown that for the first of these cases (33) the functions $\xi(x,t)$, $g(x,t)$ and $k(x,t)$ are

$$\xi = 1 - \exp\{-\tfrac{1}{4}[c_1 t + c_2]^{\frac{1}{2}(r+2)}[c_3 x + c_4]^{\frac{1}{2}(n-1)}\},$$
$$g^2 = 16[c_1 t + c_2]^{-\frac{1}{2}r}[c_3 x + c_4]^{\frac{1}{2}(1-n)}\xi,$$
$$k = 4[c_1 t + c_2]^{\frac{1}{2}r}[c_3 x + c_4]^{\frac{1}{2}(n+1)}\xi.$$

Since k, g^2 and ξ are now known, for this case, the coefficients $a(\xi)$, $b(\xi)$, $c(\xi)$, $d(\xi)$, $e(\xi)$, $h(\xi)$, $j(\xi)$, $l(\xi)$, and $m(\xi)$ given by (10), can be determined along with the coefficients of the transformed equations ((8) and (9)). Again, it is easily shown that the resulting coefficients satisfy the auxiliary equations (12).

For $\xi = 0$, the following conduction type similarity equations are obtained

$$\frac{\partial^3 f}{\partial \eta^3} + 2c_1 \eta \frac{\partial^2 f}{\partial \eta^2} - 4c_1(r+1)\frac{\partial f}{\partial \eta} + 4\theta = 0,$$
$$\frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} + 2c_1 \eta \frac{\partial \theta}{\partial \eta} - 4c_1 r\theta = 0.$$

For $\xi = 1$, the following steady-state type similarity equations are obtained

$$\frac{\partial^3 f}{\partial \eta^3} + 16c_3(n+3) f \frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta^2} - 32c_3(n+1)\left(\frac{\partial f}{\partial \eta}\right)^2 + 4\theta = 0,$$
$$\frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} + 16c_3(n+3) f \frac{\partial \theta}{\partial \eta} - 64nc_3 \theta \frac{\partial f}{\partial \eta} = 0.$$

The non-dimensional velocity in the viscous layer, in this case, is

$$u = kf' = 4[c_1 t + c_2]^{\frac{1}{2}r}[c_3 x + c_4]^{\frac{1}{2}(n+1)} f'.$$

Again this velocity should be zero for $t = 0$ at all x and at $x = 0$ for all t. These conditions are satisfied only if $c_2 = c_4 = 0$, $n > -1$ and $r > 0$, which are imposed throughout the remainder of this work.

In the case where ΔT_w varies exponentially with both distance and time (equation (34)) the functions $\xi(x,t)$, $g^2(x,t)$ and $k(x,t)$ together with the similarity equations at the end points of the range of ξ are formed in the same way. These results are also presented by Rhyne (1978). In this case, the conditions at the leading edge of the plate ($x = 0$) cannot be satisfied and the problem must be given a special physical interpretation. Thus, this case is not considered further here. In the case where ΔT_w is a linear combination of x and t, (35), it is also not possible to satisfy the physical conditions existing at the leading edge of the plate ($u = 0$ at $x = 0$ for all t and at $t = 0$ for all x). Further consideration of this case, therefore, is also omitted here.

In the preceding sections the wall-temperature distributions for which semisimilar solutions are possible have been determined and those which do not have physical significance eliminated. The scaling factors $g(x,t)$ and $k(x,t)$ were then determined for those cases which are physically realistic. Special care has been taken to construct $\xi(x,t)$ so that the infinite range of x and t in the physical problem is collapsed into a finite range in the transformed problem; i.e. $0 \leqslant \xi \leqslant 1$. In addition, we have determined the similarity forms of the equations of motion and energy which are valid at the limits of the range of ξ ($\xi = 0$ and 1). Some of these will be used as initial conditions in the computational scheme. Attention is now focused on the problem of determining solutions to (8) and (9).

3. Method of solution

The reduced equations of motion and energy, (8) and (9), subject to the boundary conditions given by (11) were solved for the cases discussed in the previous section using an implicit finite difference method. It was convenient, at the outset, to rewrite (8) and (9) as a set of equations which are second order in a new variable $W(\eta, \xi)$. In terms of W, (8) and (9) become

$$W = \frac{\partial f}{\partial \eta} = f', \tag{36}$$

$$W'' + \alpha_1 W' + \alpha_2 W + \alpha_3 = \alpha_4 \frac{\partial W}{\partial \xi}, \tag{37}$$

$$\theta'' + \beta_1 \theta' + \beta_2 \theta + \beta_3 = \beta_4 \frac{\partial \theta}{\partial \xi}, \tag{38}$$

where primes denote differentiation with respect to η and the coefficients α_1 and β_1 are given by

$$\alpha_1 = (d+e)f + \frac{b\eta}{2} + h\frac{\partial f}{\partial \xi}, \quad \alpha_2 = -df' - a, \quad \alpha_3 = j\theta, \quad \alpha_4 = c + hf',$$

$$\beta_1 = Pr\left[(d+e)f + \frac{b\eta}{2} + h\frac{\partial f}{\partial \xi}\right] = Pr\,\alpha_1, \quad \beta_2 = Pr\,[-l - mf'],$$

$$\beta_3 = 0, \quad \beta_4 = Pr\,(c + hf') = Pr\,\alpha_4.$$

In terms of W, f and θ, the boundary conditions are written

$$\left.\begin{aligned} W(\xi, 0) = 0, \quad \lim_{\eta \to \infty} W(\xi, \eta) = 0, \quad f(\xi, 0) = 0, \\ \theta(\xi, 0) = 1, \quad \lim_{\eta \to \infty} \theta(\xi, \eta) = 0. \end{aligned}\right\} \tag{39}$$

Writing the equations of motion in this form illustrates clearly the parabolic nature of the equations. If α_4 is positive, then the computation can start at $\xi = 0$, with initial profiles obtained from similar solutions, and the solution is obtained by marching forward from $\xi = 0$ to $\xi = 1$. If α_4 is negative, the solution must begin at $\xi = 1$ and develop by marching toward $\xi = 0$. A close inspection of the coefficients $c(\xi)$ and $h(\xi)$, which are important in forming α_4 indicates that for case 1, near $\xi = 0$, α_4 may be positive near the outer edge of the viscous layer and negative near the wall while sufficiently near $\xi = 1$, α_4 is negative for all values of η. For case (ii), the coefficient α_4 is positive for all values of η provided $n \geqslant 1$. For $n < 1$, however, α_4 will be positive for all η for small ξ, but for large ξ there is a region away from the wall in which α_4 will be negative. For case (iii), the coefficient α_4 is positive (or zero) for all η and ξ.

The existence of cases in which α_4 changes sign in the course of the solution causes considerable difficulty. In such cases, the problem is said to be 'singular parabolic'. Physically, the change in sign of α_4 indicates a change in the direction in which

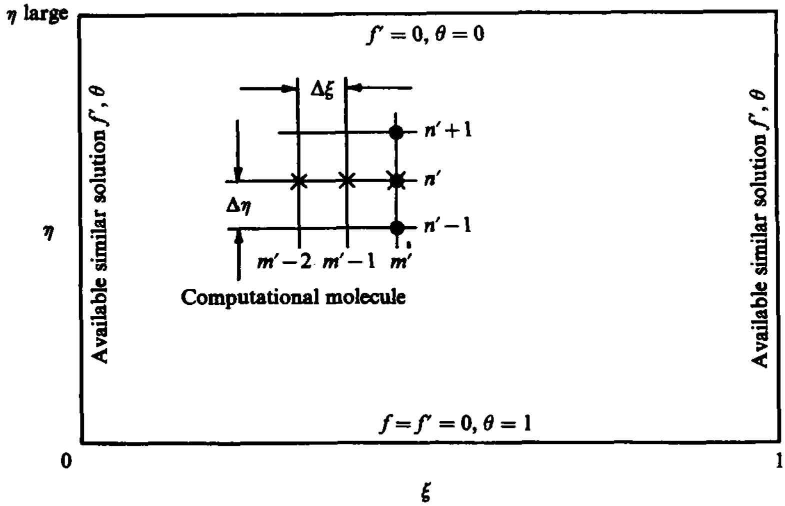

FIGURE 1. Computational region showing molecule, available similar solutions, and boundary conditions.

information is being transmitted. The classic case of this is the problem in which a finite plate is impulsively heated to a uniform temperature at time $t = 0$. This special case is imbedded in the set of solutions designated case (ii), i.e. problems in which the wall-temperature depends upon x alone. It is in fact the case of $n = 0$, i.e. uniform wall temperature. Other classic cases of such singular parabolic behaviour are the fluid dynamics of the impulsively started flat plate (Dennis 1972) and the impulsively started wedge (Williams & Rhyne 1980). These latter two problems were successfully solved using a scheme of forward differencing for a negative derivative and backward differencing when the sign of the ξ derivative is positive. However, this technique has not been successful in the classic case of free convection, which involves coupled equations, and which was attempted extensively by the present authors incorporating a solution technique presented by Carter (1974). Other authors have also attempted this problem, as mentioned previously, but all unsuccessfully and the difficulty has yet to be resolved.

In a large number of cases of transient free convection there is no change in sign of the ξ derivative during the course of the solution. Also, in those cases where a change in sign does occur, often the portion of the solution up to the change in sign is most important. Computations for these cases are presented here. Three-point backward ξ differences were used, starting at $\xi = 0$ or $\xi = 1$, depending on the sign of α_4. The computational molecule is shown in figure 1. Using central differences in η, the discretized equations become

$$A_{n'} W_{n'+1,\, m'} + B_{n'} W_{n',\, m'} + C_{n'} W_{n'-1,\, m'} = R_{n'}, \tag{40}$$

$$AT_{n'} \theta_{n'+1,\, m'} + BT_{n'} \theta_{n',\, m'} + CT_{n'} \theta_{n'-1,\, m'} = RT_{n'}, \tag{41}$$

where

$$A_{n'} = \left[\frac{1}{\Delta\eta^2} + \frac{\alpha_1}{2\Delta\eta}\right]_{n'}, \quad B_{n'} = \left[-\frac{2}{\Delta\eta^2} + \frac{3\alpha_4}{2\Delta\xi} + \alpha_2\right]_{n'},$$

$$C_{n'} = \left[\frac{1}{\Delta\eta^2} - \frac{\alpha_1}{2\Delta\eta}\right]_{n'}, \quad R_{n'} = \left[\frac{2\alpha_4}{\Delta\xi} W_{m'-1} - \frac{\alpha_4}{2\Delta\xi} W_{m'-2} - \alpha_3\right]_{n'},$$

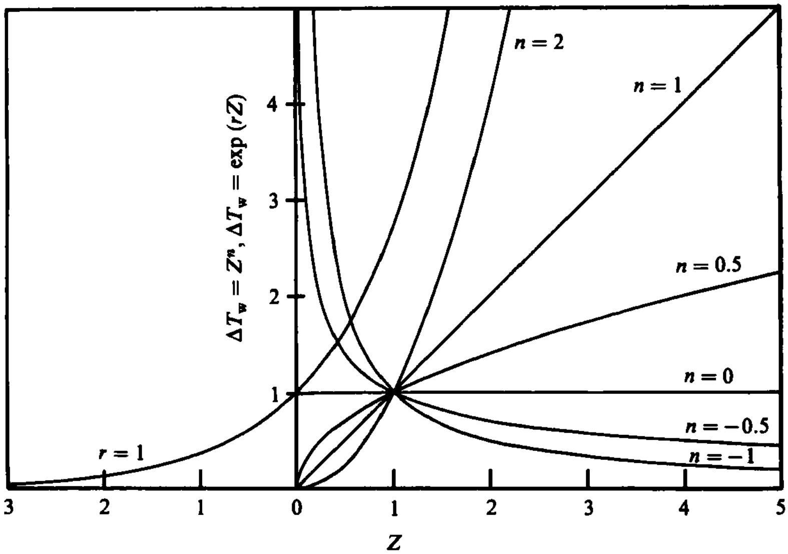

FIGURE 2. Illustration of two fundamental variations of wall temperature with position, $Z = x$, and time, $Z = t$.

$$AT_{n'} = \left[\frac{1}{\Delta\eta^2} + 2\Delta\eta\right]_{n'}, \quad BT_{n'} = \left[-\frac{2}{\Delta\eta^2} + \frac{3\alpha_4}{2\Delta\xi}\right]_{n'},$$

$$CT_{n'} = \left[\frac{1}{\Delta\eta^2} - \frac{\alpha_1}{2\Delta\eta}\right]_{n'}, \quad RT_{n'} = \left[\frac{2\alpha_4}{\Delta\xi}\theta_{m'-1} - \frac{\alpha_4}{2\Delta\xi}\theta_{m'-2}\right]_{n'}.$$

Equations (40) and (41) were written as a system of tridiagonal matrix equations at each ξ station, and solved by a Thomas algorithm. The procedure was to assume η distributions for W and θ, obtain f from (36), and then compute the nonlinear coefficients. This step effectively linearizes and uncouples the equations. The coefficients were then used in the Thomas algorithm to generate a new W and θ profile, and the process repeated until convergence. Moving to the next ξ station, the entire process was repeated and the solution marched through ξ range.

4. Results

Solutions were developed numerically after the application of the semisimilar solution technique, with results generated for the following cases of wall-temperature variation

$$\Delta T_w = [c_1 t + c_2]^r, \tag{42}$$

$$\Delta T_w = k_1 e^{k_2 t}, \tag{43}$$

$$\Delta T_w = [c_3 x + c_4]^n, \tag{44}$$

$$\Delta T_w = k_3 e^{k_4 x}, \tag{45}$$

$$\Delta T_w = [c_1 t + c_2]^r [c_3 x + c_4]^n, \tag{46}$$

$$\Delta T_w = k_1 e^{k_2 t + k_4 x}. \tag{47}$$

Since the constants c_2 and c_4 have no real effect on the solutions, they were taken as zero. The constants c_1, c_3, k_1, k_2, k_3 and k_4 affect the solutions by degree only and not in character if they are positive and non-zero. These constants were set equal to unity, except in noted cases where the zero value was investigated. Figure 2 shows a sample of the above wall-temperature profiles in terms of a variable Z which may be replaced with x or t. The wall temperatures varying with both x and t will, of course, be three-dimensional combinations of these profiles.

The solutions presented are given primarily in terms of the conventional dimensionless ratio involving the local heat transfer coefficient versus the appropriate ξ. This heat transfer grouping is defined as

$$\frac{Nu_{xt}}{Gr_{xt}^{\frac{1}{4}}} = -\frac{x^{\frac{1}{4}}}{\Delta T_w^{\frac{1}{4}}} \frac{\theta'(\eta = 0)}{g}, \tag{48}$$

where
$$Nu_{xt} = \frac{h_c \bar{x}}{k} = \frac{-\bar{q}(\bar{x}, \bar{t})}{k(\bar{T}_w - \bar{T}_\infty)} \bar{x}, \tag{49}$$

and
$$Gr_{xt} = \beta g_0 \bar{x}^3 (\bar{T}_w - \bar{T}_\infty)/\nu^2. \tag{50}$$

All solutions presented are for Prandtl number equal to one except for one example showing the effect of different Prandtl numbers.

4.1. *Wall temperature varying with time*

For all cases of the time varying wall temperature, (42) and (43), the solutions were found to follow the analytic one-dimensional solutions for the doubly infinite flat plate given by Menold & Yang (1962), until the coefficient of the ξ derivative term, α_4, changes sign. At this change of sign the solutions diverged. This change of sign indicates the entrance of an x dependence in the physical problem; that is, it indicates the penetration distance of the leading-edge effect. Physically, the coefficient changes sign as the fluid from the region of the leading edge first reaches a given x position along the plate.

Figure 3 shows the above result in the heat transfer grouping for several values of r. The termination points are compared with the penetration distance computations by Mizukami (1977), and thus tend to verify the present results. The leading-edge effect introduces a singularity into the boundary-layer equations, and all attempts to solve past this point failed. Note that at least for values of $r \leqslant 0$, there will occur a minimum in the heat transfer coefficient. Also note that $r = 0$ is the impulsively heated constant-temperature flat plate.

Physically, the following interpretation is given for this particular problem. The plate, initially at ambient temperature, begins to be heated with a wall temperature varying as a power of time. For short times or large x, the fluid near the plate behaves as if the plate were infinite in extent and the one-dimensional conduction solution applies. After a time, fluid from the leading edge reaches a given x position on the plate, introducing an x dependence to the flow. The fluid thus departs from the one-dimensional conduction solution. Near $x = 0$ or for long times, the flow about the plate is found to approach that of the steady-state constant-wall-temperature free convection case, regardless as to the exponent r. The lower limit on the wall-temperature exponent, r, for physically meaningful solutions, is not clear. However, no results were computed for $r < 0$.

Solutions for the exponential wall temperature, (43), were found to behave in the same manner as the previous case. The heating begins, however, at $t = -\infty$. The results are given in figure 4. Finally, the cases of $c_1 = 0$ in (42) and $k_2 = 0$ in (43) were checked and found to be continuations of the steady solution valid at $\xi = 0$.

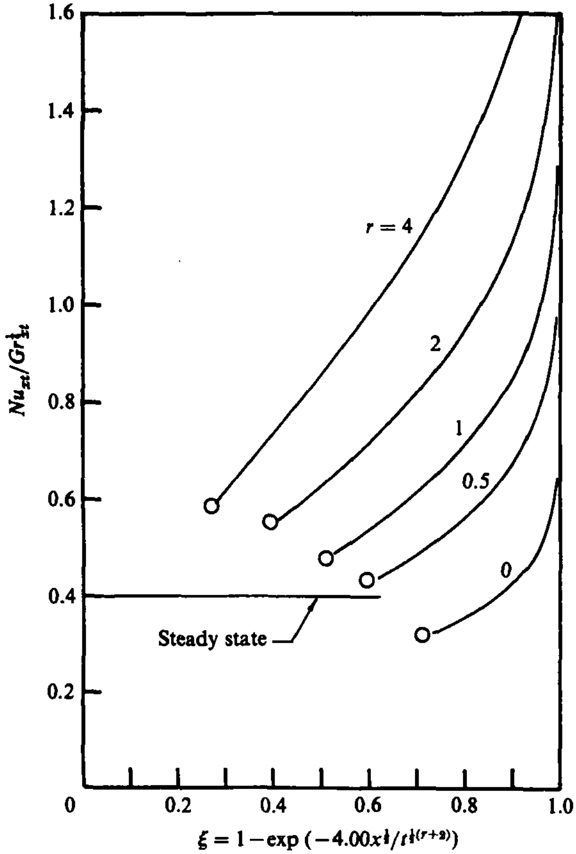

FIGURE 3.

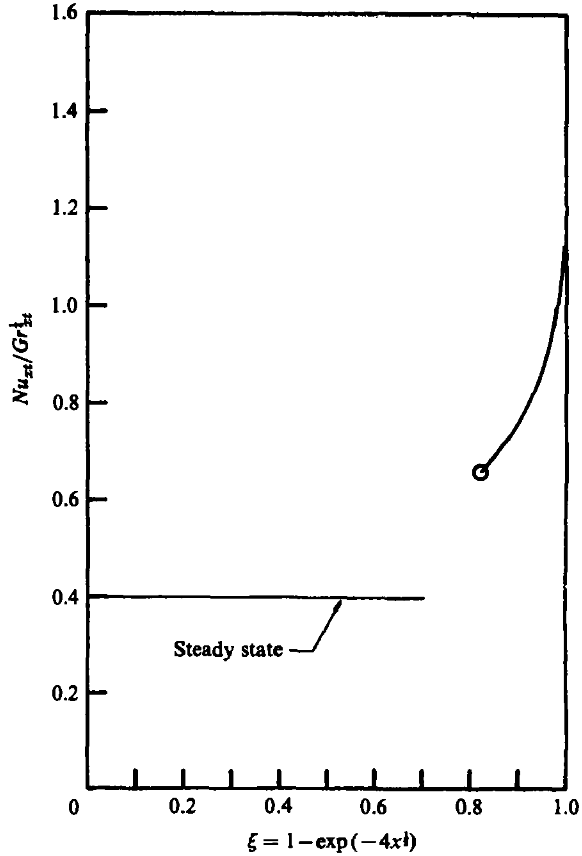

FIGURE 4.

FIGURE 3. Variations of local heat transfer grouping for $T_w = t^r$, showing limits of pure conduction and overshoot of steady state, ○, Mitzukami

FIGURE 4. Solution for wall temperature varying exponentially with time, $\Delta T_w = e^t$, showing the limit-of-pure-conduction without an overshoot. ○, Mitzukami.

4.2. *Wall temperature varying with position*

The impulsively heated plate is of considerable interest because the transient nature of the resulting free convection problem is displayed independently of the effects of varying wall temperature. Wall temperature given by (44) was considered first with results, in terms of the heat transfer grouping, given in figure 5 for various values of n. For values of $n \geqslant 1$, complete solutions were obtained. These are believed to be the first accurate numerical solutions for the full transient free convection problem. All solutions for any value of n were found to exhibit a minimum in the heat transfer grouping, and the complete solutions clearly confirmed the existence of this minimum. Interestingly, the magnitude of this overshoot of the heat transfer coefficient reaches a minimum at $n = 1$.

Only partial solutions were computed for $n < 1$. As for the previous case, the solution is found until the coefficient of the ξ derivative term changes sign, thus causing the essential singularity. For $n = 0$ the solution for the impulsively heated constant temperature plate is recovered and follows the pure conductive solution exactly until the coefficient of the ξ derivative changes sign, a point often referred to as the 'limit

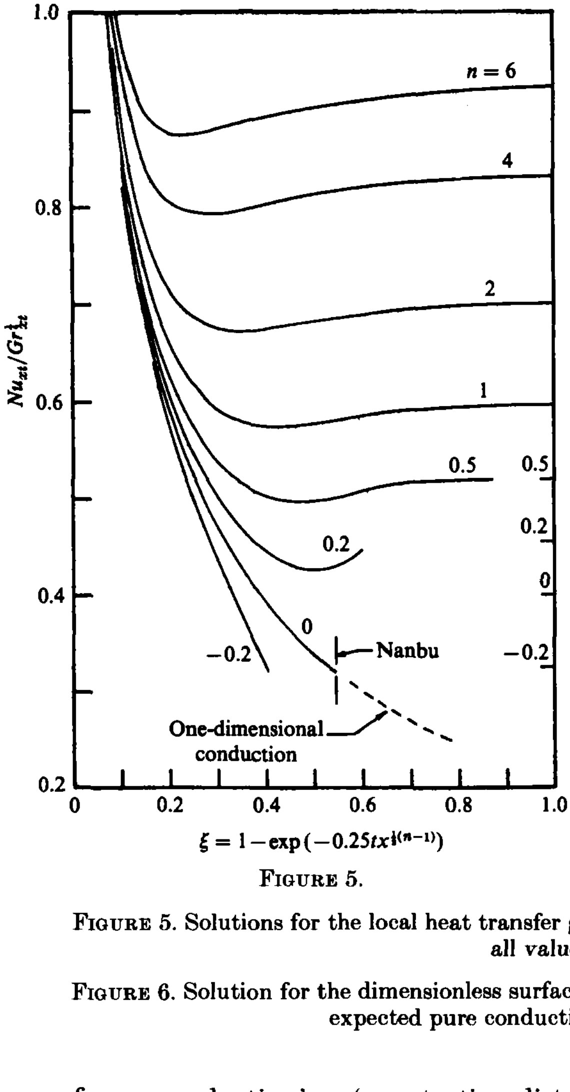

FIGURE 5.

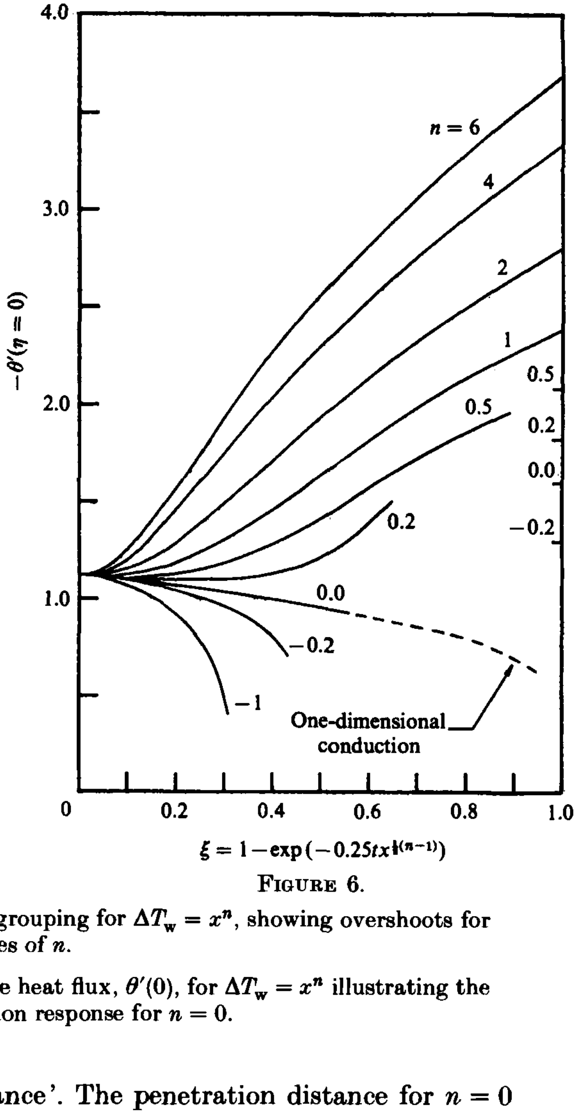

FIGURE 6.

FIGURE 5. Solutions for the local heat transfer grouping for $\Delta T_w = x^n$, showing overshoots for all values of n.

FIGURE 6. Solution for the dimensionless surface heat flux, $\theta'(0)$, for $\Delta T_w = x^n$ illustrating the expected pure conduction response for $n = 0$.

of pure conduction' or 'penetration distance'. The penetration distance for $n = 0$ predicted by Nanbu (1971) is also shown in figure 5 and is seen to coincide with the end of the calculated solution. The continuation of the one-dimensional conduction solution is shown by the dotted line in the figure.

For other values of n, the conduction limit takes on a different meaning. Previously, since the wall temperature did not vary with x, i.e. $\Delta T_w(t)$, the leading-edge effect introduced the x dependence into the physical problem and thus defined the penetration distance and the limit-of-pure-conduction to be the same. This cannot be true, however, for the present case, except for $n = 0$. For $n \neq 0$ an x dependence enters the problem immediately at all positions on the plate, through the wall temperature. Indeed, for $n \neq 0$ the solutions immediately depart from the one-dimensional conduction solution found to be valid at $\xi = 0$. A pure conductive phase of the transport, therefore, simply does not exist and the point of singularity can only be interpreted as the penetration distance of the leading-edge effect.

Figures 6 and 7 more clearly show the departure from the one-dimensional conduction solution for $n \neq 0$, through both $\theta'(\eta = 0)$ and the dimensionless wall

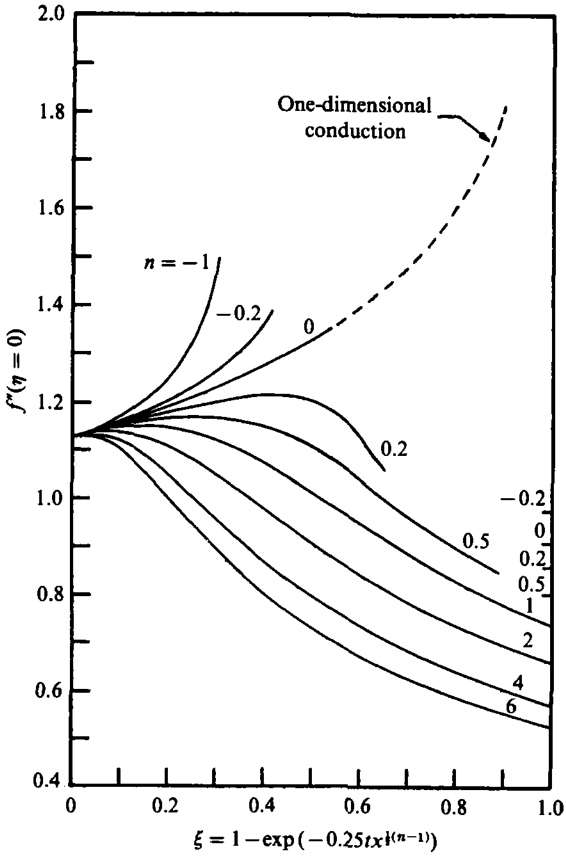

FIGURE 7.

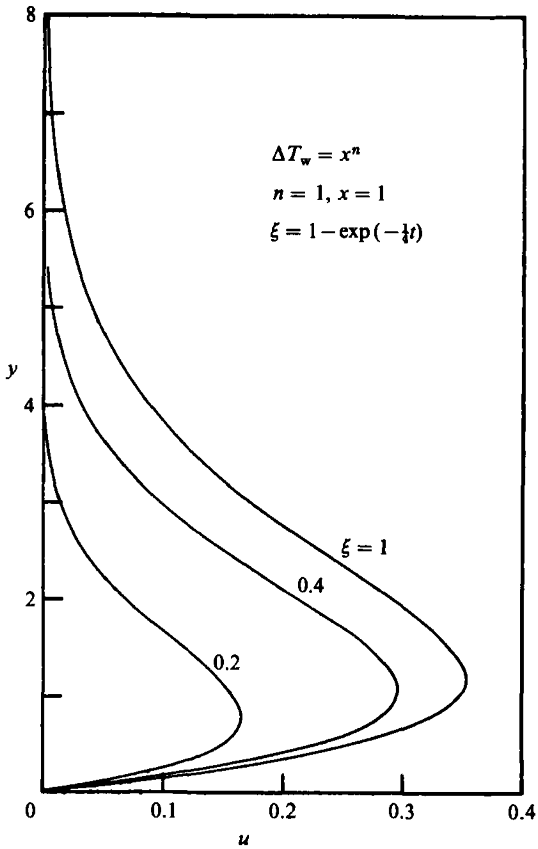

FIGURE 8.

FIGURE 7. Variation of the dimensionless wall shear, $f''(0)$, with time for $\Delta T_w = x^n$.

FIGURE 8. Velocity profiles for $\Delta T_w = x (n = 1)$, showing a normal transient to a steady state.

shear $f''(\eta = 0)$. Again, the continuation of the one-dimensional conduction solution is shown as a dotted line in both figures. Since f' is a factor in the coefficient of the leading ξ derivative, α_4, and since analytic solutions are not known for $n \neq 0$, the penetration distance of the leading-edge effect can only be determined numerically, as the point at which this coefficient vanishes.

Clearly, for $n \geqslant 1$ the coefficient (α_4) is positive for all ξ. This indicates that no leading-edge signal is propagated for $n \geqslant 1$. Evidently the propagation of a leading-edge disturbance in the physical problem or the introduction of a singularity in the mathematical model is dependent upon the x derivative of the wall-temperature distribution at $x = 0$. If $n \geqslant 1$ the slope of the wall temperature at $x = 0$ is finite or zero, no singularity is introduced, and no disturbance is propagated along the plate. For $n < 1$ the slope is infinite at $x = 0$ and a singularity is introduced in the mathematical equations. It is noted that for the purely time-dependent wall temperatures considered in the previous section, there is a step change in wall temperature at $x = 0$ and thus a singularity is introduced in all cases.

Referring again to figure 5, an interesting observation is made for $n = 0.5$. The point at which the solution fails is past the position at which the flow has essentially

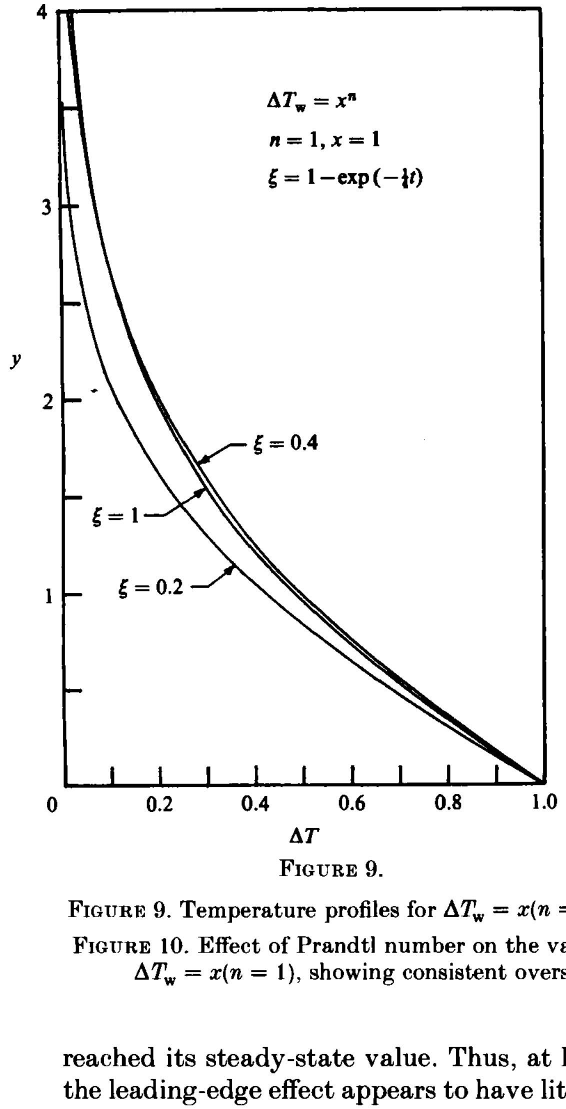

FIGURE 9.

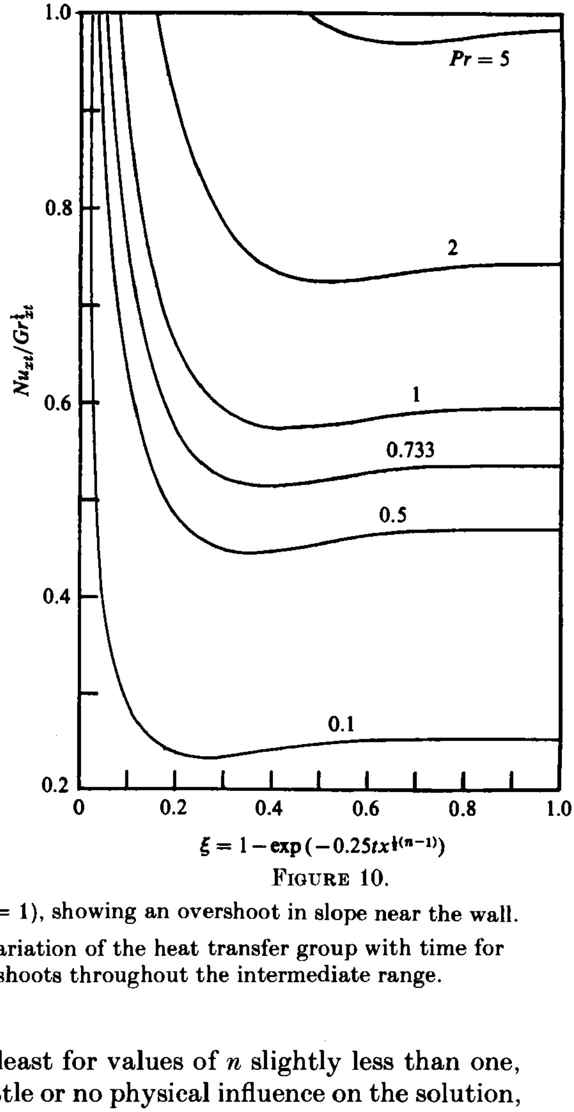

FIGURE 10.

FIGURE 9. Temperature profiles for $\Delta T_w = x(n = 1)$, showing an overshoot in slope near the wall.

FIGURE 10. Effect of Prandtl number on the variation of the heat transfer group with time for $\Delta T_w = x(n = 1)$, showing consistent overshoots throughout the intermediate range.

reached its steady-state value. Thus, at least for values of n slightly less than one, the leading-edge effect appears to have little or no physical influence on the solution, and certainly does not coincide with the minimum as has been suggested.

The steady-state solutions valid at $\xi = 1$ and shown in figures 5, 6 and 7 are given by (28) and (29). The solutions to these equations were first numerically obtained by Sparrow & Gregg (1958). The heat transfer coefficients were compared with Sparrow & Gregg's published results and were found to match. The solution for the case of $c_3 = 0$ in (44) is found to give a continuation of the one-dimensional conduction solution.

Velocity profiles for $n = 1$ and several values of ξ are given in figure 8. No overshoot of the velocity profiles or boundary-layer thickness is found in this case. Figure 9 shows the corresponding temperature profiles. Here it is seen that the slope of the temperature at $\xi = 0.4$ exceeds the steady state value from the wall through most of the boundary layer, and thus gives the overshoot observed in the heat transfer grouping.

The case of $n = 1$ is again used in figure 10 to show the effect of Prandtl number variation. Overshoots of the heat transfer are found for all Prandtl numbers computed. As the Prandtl number decreases, the steady state is reached earlier, reflecting the quicker thermal response of lower Prandtl number fluids.

An interesting result can be seen for the limiting case of $\xi = 1$ when $n > 1$. The limit of $\xi \to 1$ corresponds to the two cases of x finite while $t \to \infty$, and t finite while $x \to \infty$. Steady state is represented by the former, and the downstream asymptote by the latter. In going to the steady-state limit a solution is obtained in which $\xi = 1$, and $\eta = \frac{1}{4}yx^{\frac{1}{4}(n-1)}$. Hence, both θ and f' become functions which are independent of the physical time, t, and dependent only upon y and x, as expected. In going to the downstream limit, however, it must be kept in mind that at physical $x = \infty$, $\Delta T_w = \infty$, and $\eta = \infty$, which obviously has little physical significance. Hence, it is important that in going to this limit one considers a large but bounded x. For this case, it is seen that $\xi = 1 - \exp(-\epsilon)$ and $\eta = (y/2t^{\frac{1}{2}})(\epsilon \exp(\epsilon)/(\exp(\epsilon) - 1))^{\frac{1}{2}}$, where $\epsilon = \frac{1}{4}tx^{\frac{1}{2}(n-1)}$. Hence, at large ϵ it is seen that

$$\xi = 1 + O(\exp(-\epsilon)),$$
$$\eta = (\tfrac{1}{4}yx^{\frac{1}{4}(n-1)})(1 + O(\exp(-\epsilon)),$$

and that ξ and η both possess an unsteady behaviour for all x which weakens as x becomes large. For $\epsilon \geqslant 7$, ξ is within approximately 0.1 % of unity and η is within 0.1 % of its steady-state behaviour. Hence, $\epsilon = 7$ might be considered as a criterion to use in defining the $x-t$ envelope of steady flow. At larger values of x (for $n > 1$) the time, t, required for steady state is smaller, indicating that downstream locations establish a steady state more quickly than upstream locations. This is consistent with the shape of the ΔT_w distribution in x for $n > 1$.

The range of n for which physically significant free convection problems result is now considered. There appears to be no limitation on the values of positive n. Sparrow & Gregg (1958) have considered the possible values of negative n for the steady-state case. They found that for $n < -0.6$, there is an infinite source of energy in the fluid at the leading edge. The heat transfer coefficient becomes negative, indicating heat transfer to the wall even though $\overline{T}_w > \overline{T}_\infty$. This is, of course, not physically realistic. Additionally, for $n < -0.8$, no steady-state solution is found. Figure 5 illustrates these results for the transient case. Therefore, $n = -0.6$ is considered the lower limit for physically realistic results.

The exponential wall-temperature variation, (45), is found to behave in a similar manner as the power of x case above. As explained previously, this case corresponds to a plate whose leading edge lies at $x = -\infty$. The x derivative of the wall temperature in this case is zero at $x = -\infty$, and thus no leading-edge singularity is introduced and a complete solution is computed. The heat transfer coefficient, however, is not properly formulated for this case. Therefore, only $-\theta'(\eta = 0)$ is plotted in figure 11. Temperature profiles are shown in figure 12 which verify an overshoot of the temperature in the boundary layer before the steady state is reached.

4.3. *Wall temperature varying with position and time*

Wall temperature described by (46) is considered first. Any number of combinations of n and r could be used in this case. All wall temperatures whose x derivatives are not infinite at $x = 0$ yield complete solutions. As before, this is due to the introduction of a singularity in the equations if the slope of the wall temperature is infinite at $x = 0$. The results presented here are restricted to the particular case of $n = 1$ for various r values which are representative of the numerous complete solutions that can be obtained.

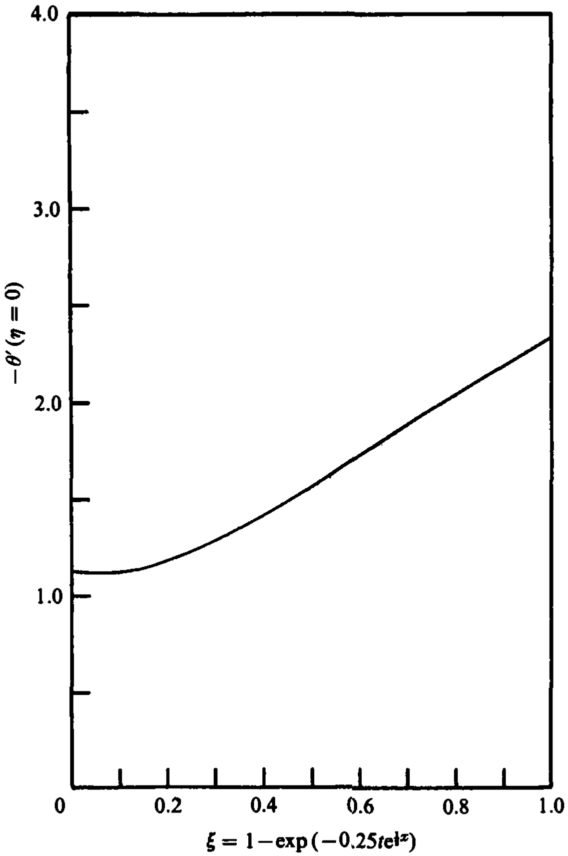

FIGURE 11.

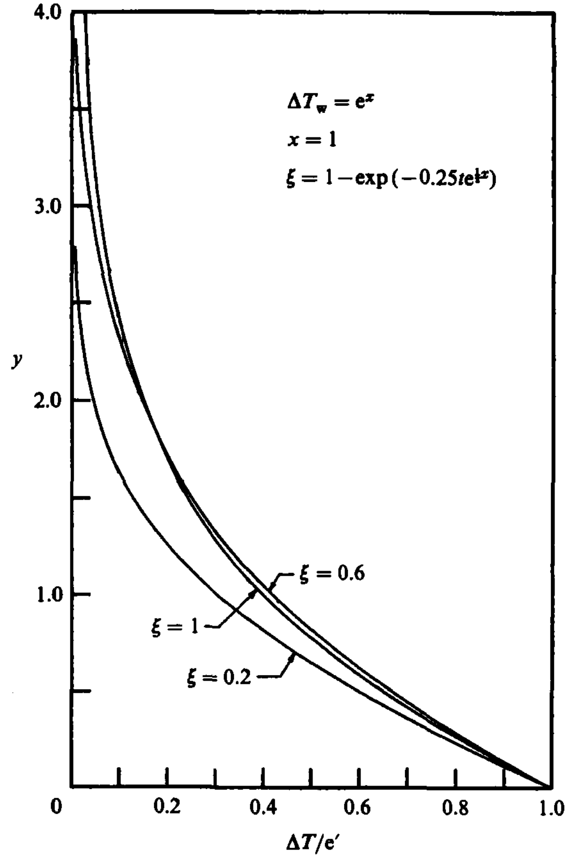

FIGURE 12.

FIGURE 11. Complete solution for the dimensionless wall heat flux, $\theta'(0)$, for the case $\Delta T_w = e^x$, obtainable because of the absence of a leading-edge singularity at $x = -\infty$.

FIGURE 12. Temperature profiles for $\Delta T_w = e^x$, showing an overshoot in the boundary layer before steady state is reached even though no singularity is encountered

Figure 13 gives the heat transfer coefficient for various values of r. ξ is independent of x for $n = 1$. These unsteady heat transfer profiles then occur simultaneously for all values of x. The heat transfer coefficient is seen to overshoot the long time value if $-0.5 < r < 0$. For $r = -0.5$ there is no heat transfer to the fluid initially. For $r < -0.5$ the heat transfer coefficient is negative initially, indicating a physically unrealistic situation.

Figure 14 shows clearly the solutions valid at the endpoints of ξ. For $\xi = 0$ or $t = 0$ the one-dimensional conduction equations for time varying wall temperatures are valid and the solutions are independent of n. At $\xi = 1$ or $t = \infty$ clearly steady-state type equations apply and solutions are independent of r.

The case of the exponential wall temperature of (47) corresponds physically to a plate with the leading edge at $x = -\infty$. The x derivative of the wall temperature at $x = 0$ is zero, and thus a complete solution is obtained. Again the heat transfer grouping is not properly defined for this case and thus only $\theta'(\eta = 0)$ is plotted in figure 15. This complete unsteady solution and the ones found in the previous paragraphs are the only such solutions in the literature, to the author's knowledge.

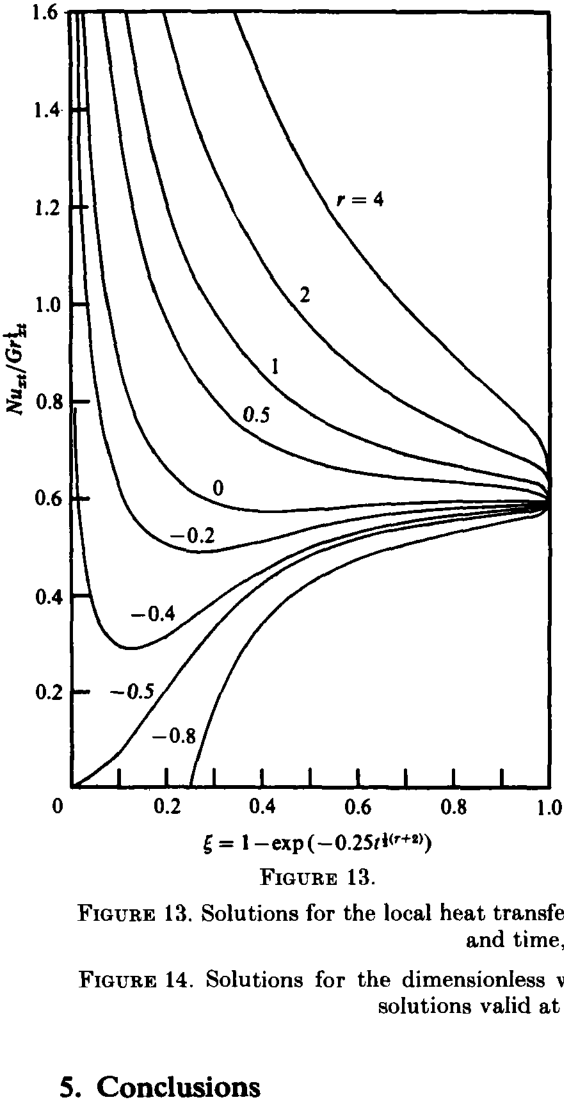

FIGURE 13.

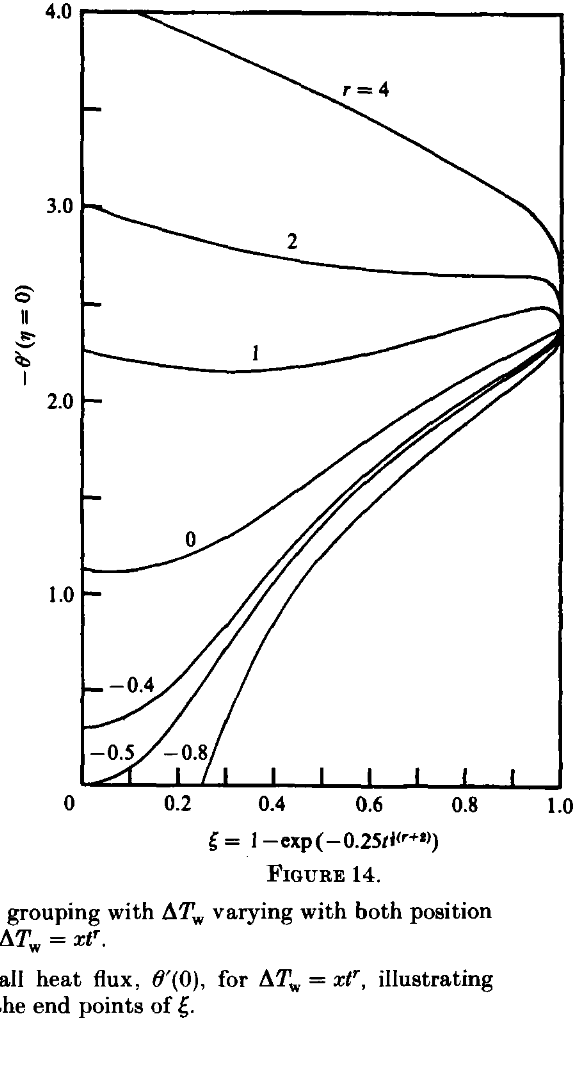

FIGURE 14.

FIGURE 13. Solutions for the local heat transfer grouping with ΔT_w varying with both position and time, $\Delta T_w = xt^r$.

FIGURE 14. Solutions for the dimensionless wall heat flux, $\theta'(0)$, for $\Delta T_w = xt^r$, illustrating solutions valid at the end points of ξ.

5. Conclusions

The unsteady free convection from a vertical flat plate has been analysed using the method of semisimilar solutions, thus avoiding the numerical pitfalls which have plagued other investigators. An analytical procedure has been developed which requires only state-of-the-art numerical techniques, while restricting the form of the allowable boundary conditions. Nevertheless, it is shown that applying such a technique to this free convection problem not only provides much useful information about the physics of the flow phenomena and the classical unsteady flow anomalies, but also provides solutions which can function as important benchmarks in future computational approaches. The lack of such data is perhaps the reason completely numerical procedures have failed in the past.

In the results it is shown that concepts such as the 'limit-of-pure-conduction' and 'penetration distance' must be used very carefully when variable wall temperatures are involved. The presumption that has been made for years is that both of these are the same only cast in terms of different variables (time for the 'limit' and distance for the 'penetration'), and both correspond to the occurrence of the 'leading-edge

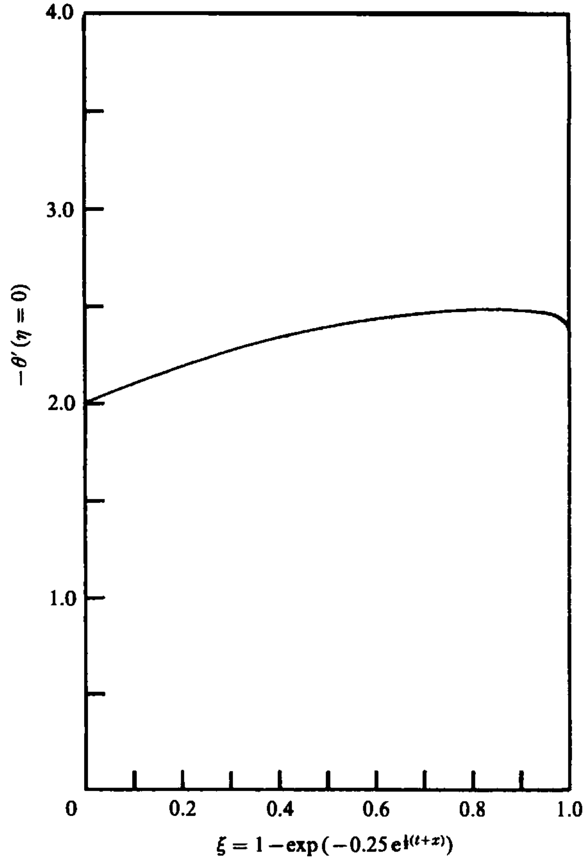

FIGURE 15. Solution for the dimensionless wall heat flux, $\theta'(0)$, for the case $\Delta T_w = e^{t+x}$.

singularity'. We now know that this is true only for the very special case of constant and uniform wall temperature. In fact, we have shown that under some circumstances the 'overshoot' which has classically been related to the 'limit-of-pure-conduction' and the 'penetration distance', and interpreted in light of the singularity, occurs in cases where no singularity is encountered. The problem is therefore much more complex than previously believed.

The next step in studying this unsteady free convection should be to remove the boundary-layer approximations and attempt a computational solution in full formulation. Many of the numerical difficulties which we and others have encountered are believed to be due to the essential singularity in the equations which occurs, in principle, because of the exclusion of streamwise second derivatives. With the checkpoints presented in this paper, such an approach might be successful in filling in information which is not now available.

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